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A new technique to solve generalized Caputo type fractional differential equations with the example of computer virus model

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Abstract. In this research work, we proposed a new fractional numerical algorithm to obtain the exact solutions of generalized fractional-order differential equations in Caputo sense of order $0 < \theta \leq 1$. For finding the exact solutions by the proposed technique we used the solutions of integer-order differential equations. Generalization of the proposed scheme to finite systems is also introduced. At the last, we gave

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some numerical simulations of some specific equations along with the solution of a computer virus model to illustrate the applications of our results.

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1 Introduction

Current days, Fractional Calculus, also known as non-integer calculus, is the mostly used phenomena in the different fields of science and engineering. Day by day so many different types of fractional order derivatives have been proposed or investigated by researchers along which numerical methods to solve non-integer order (fractional) differential equations (FDEs). So many techniques to find the numerical solution of FDEs have been studied in previous works [10, 19]. In most of the cases, in such methods, either the solutions of a classical differential equation is the translation of the given non-integer order differential equations or the series diffusions in the neighbourhood of the initial constraints are utilized. These numerical methods are playing a very important role in the study of complex dynamics. Also, fractional (non-integer) order differential equations (FDEs) are very useful in mathematical modelling. So many non-integer order derivatives have been used in various parts of engineering and science [7, 10, 17, 19, 20]. Caputo, Atangana-Baleanu-Caputo (AB), and Caputo-Fabrizio (CF) derivatives are the well known fractional derivative operators. Caputo derivative has derived with singular type non-local kernel (or power law type), Caputo-Fabrizio with exponentially decay-type (or non-singular kernel) and AB derivative has given with Mittag-Leffler kernel memory. Recently *Odibat et al.* [16] has introduced a new modified-version of Caputo-type variable-order derivative with the modified P-C scheme. The existence proof of a unique solution of the generalized Caputo type FDEs is proved by *katugampola et al.* in [9]. Currently so many applications of different fractional derivatives have been come in epidemiology [4, 7, 11, 12, 13, 15]. In [8], the authors proposed a study on tight-bounds for the path-factors existence in the parameter settings of network vulnerability. Recently, a number of mathematicians have analysed some important new algorithms for finding the solutions of non-linear FDEs. In [6], a new scheme to simulate Atangana-Baleanu type non-linear volterra integro-differential equations is produced. In [21], one more method for the FDEs based on Genocchi polynomials has mentioned. *Ganji et al.* in [5] suggested a new method to solve AB-type multi-variable order differential equations. The complex dynamics which can not be studied by classical derivatives, can be studied by these non-classical derivatives more clearly. Still, there are many drawbacks in the non-classical calculus. In several cases, the solutions existence for many FDEs can not be smoothly described. Lately, an existence analysis for infinite-coefficient symmetric integro-DEs in Caputo-Fabrizio form is given by *Baleanu* in [1].

In this paper, we generalize the numerical technique introduced by *Demirci et al.* in [3]. We utilize a transformation in the equivalent non-integer order Volterra integral equation (VIE) of given FDE and enlist its exact solution in the form of the solution of an classical-order differential equation in the form of generalized Caputo type non-classical derivatives. We explained some examples to show the applications of the given scheme clearly. The paper is distributed as follows. In Section 2, we remind some specific definitions of variable-order derivatives. In Section 3, we review the original method. Section 4 is devoted to the description of the main results. Some examples with the solution of a computer virus epidemic model to show the applications of the given method are given in section 5. A conclusion finish the paper.

2 Preliminaries

Here we recall some necessary definitions of the fractional (or non-integer order) derivatives.

Definition 2.1. [19] The Riemann and Liouville (R-L) non-classical derivative of order $\theta > 0$ of a mapping $X : (0, \infty) \to \mathbb{R}$ is formulated by

$$D_{\eta}^{\theta} X(\eta) = \left(\frac{d}{d\eta}\right)^{n} \frac{1}{\Gamma(n-\theta)} \int_{0}^{\eta} (\eta - \xi)^{n-\theta-1} X(\xi) d\xi, \tag{1}$$

where $n = [\theta] + 1$ and $[\theta]$ is the integer-part of θ.

Definition 2.2. [19] The Caputo-type non-integer order derivative of order $\theta > 0$ of a mapping $X : (0, \infty) \to \mathbb{R}$ is described by

$$D_{\eta}^{\theta} X\,(\eta) = \frac{1}{\Gamma(k-\theta)} \int_{0}^{\eta} (\eta - \xi)^{k-\theta-1} X^{k}(\xi) d\xi, \tag{2}$$

where $k = [\theta] + 1$ and $[\theta]$ is the integer-part of θ.

Definition 2.3. [9] The generalized Riemann-type non-classical derivative operator, ${}^{R}D_{c+}^{\theta,\rho}$, of order $\theta > 0$ is given as:

$$({}^{R}D_{c+}^{\theta,\rho} X)(\eta) = \frac{\rho^{\theta-n+1}}{\Gamma(n-\theta)} \left(\eta^{1-\rho} \frac{d}{d\eta}\right)^{n} \int_{c}^{\eta} s^{\rho-1} (\eta^{\rho} - s^{\rho})^{n-\theta-1} X(s) ds, \;\; \eta > c, \tag{3}$$

where $c \geq 0, \;\; \rho > 0, \;\; \& \; n - 1 < \theta \leq n.$

Definition 2.4. [9] The generalized-version of Caputo-type non-classical derivative, ${}^{C}D_{c+}^{\theta,\rho}$, of order $\theta > 0$ is given as:

$$({}^{C}D_{c+}^{\theta,\rho} X)(\eta) = \left({}^{R}D_{c+}^{\theta,\rho}\left[X(x) - \sum_{k=0}^{n-1} \frac{X^{(k)}(c)}{k!}(x-c)^{k}\right]\right)(\eta), \quad \eta > c, \tag{4}$$

where $c \geq 0, \;\; \rho > 0, \;\; \& \; n = \lceil \theta \rceil.$

Definition 2.5. [16] The version of new modified generalized Caputo non-integer order derivative, $D_{c+}^{\theta,\rho}$, of order $\theta > 0$ is described as:

$$(D_{c+}^{\theta,\rho} X)(\eta) = \frac{\rho^{\theta-n+1}}{\Gamma(n-\theta)} \int_{c}^{\eta} s^{\rho-1} (\eta^{\rho} - s^{\rho})^{n-\theta-1} \left(s^{1-\rho} \frac{d}{ds}\right)^{n} X(s) ds, \;\; \eta > c, \tag{5}$$

where $c \geq 0, \;\; \rho > 0, \;\; \& \; n - 1 < \theta \leq n.$

3 The solution method in Caputo sense

In this portion of the study, we review the numerical algorithm proposed in [3]. This method is rooted on converting the non-integer order systems to a classical (integer-order) systems and finding the solution of the non-integer order systems in the form of the solution of the classical systems. Let us remind the initial value problem (IVP)

$$\begin{aligned} &{}^{C}D_t^{\theta}\zeta(t) = \mathcal{G}(t, \zeta(t)), \\ &\zeta(0) = \zeta_0, \end{aligned} \tag{6}$$

where $\mathcal{G} \in C([0, T] \times \mathbb{R}, \mathbb{R}),\ 0 < \theta < 1$.

Since $\mathcal{G}$ is considered as a continuous mapping or function, so each solution of the IVP (6) is also satisfy the given Volterra fractional integral equation (VFIE):

$$\zeta(t) = \zeta_0 + \frac{1}{\Gamma(\theta)} \int_0^t (t-\chi)^{\theta-1} \mathcal{G}(\chi, \zeta(\chi)) d\chi, \quad t \in [0, T]. \tag{7}$$

Additionally, every solution of (7) is also satisfy the IVP (6). We find that IVP(6) is similar to the IVP

$$\begin{aligned} &{}^{C}D_t^{\theta}(\zeta(t) - \zeta_0) = \mathcal{G}(t, \zeta(t)), \\ &\zeta(0) = \zeta_0. \end{aligned}$$

Theorem 3.1. *[14](existence) Assume that $\mathcal{G} \in C[R_0, R]$ where $R_0 = \{(t, \zeta) : 0 \le t \le m \text{ and } |\zeta - \zeta_0| \le b\}$ and fix $|\mathcal{G}(t, \zeta)| \le M$ on R_0. Then at least one solution for the IVP (6) exists on $0 \le t \le \delta$, where $\delta = \min\big(m, \big[\frac{b}{M}\Gamma(\theta+1)\big]^{\frac{1}{\theta}}\big), 0 < \theta < 1$.*

Theorem 3.2. *Consider the IVP proposed by (6). Let*

$$\phi(\nu, \zeta_*(\nu)) = \mathcal{G}(t - (t^{\theta} - \nu\Gamma(\theta+1))^{\frac{1}{\theta}}, \zeta(t - (t^{\theta} - \nu\Gamma(\theta+1))^{\frac{1}{\theta}})),$$

and suppose that the Theorem 3.1 hold. Then, a solution of (6), $\zeta(t)$, is established by

$$\zeta(t) = \zeta_*(t^{\theta}/\Gamma(\theta+1)),$$

where $\zeta_*(\nu)$ is a solution of the classical differential equations

$$\frac{d\zeta_*(\nu)}{d\nu} = \phi(\nu, \zeta_*(\nu)), \tag{8}$$

with the initial conditions

$$\zeta_*(0) = \zeta_0. \tag{9}$$

Proof. According to the Theorem 3.1, the solution of the (6) is exists. If $\zeta(t)$ is a solution of (12) then, it is also a satisfy the (7). Let $\tau = t - (t^\theta - \nu\Gamma(\theta+1))^{1/\theta}$. So, VFIE (7) can be expressed as

$$\zeta(t) = \zeta_0 + \int_0^{t^\theta/\Gamma(\theta+1)} \mathcal{G}(t - (t^\theta - \nu\Gamma(\theta+1))^{1/\theta}, \zeta(t - (t^\theta - \nu\Gamma(\theta+1))^{1/\theta}))d\nu$$
$$= \zeta_0 + \int_0^{t^\theta/\Gamma(\theta+1)} \phi(\nu, \zeta_*(\nu))d\nu. \tag{10}$$

Also every solution of (8)-(9) is a solution of the VFIE written below and vice versa.

$$\zeta_*(\nu) = \zeta_0 + \int_0^\nu \phi(s, \zeta_*(s))ds, \quad 0 \le \nu \le a^\theta/\Gamma(\theta+1). \tag{11}$$

Since $0 \le t^\theta/\Gamma(\theta+1) \le a^\theta/\Gamma(\theta+1)$, the right-hand part of equation (10) is equal to $\zeta_*(t^\theta/\Gamma(\theta+1))$. □

4 Main Results in new generalised Caputo sense

After successfully reviewed the above method in the Caputo sense, now we do our main simulations with the generalise form of Caputo differential operator. Let us adopt the IVP

$${}^C D_t^{\theta,\rho}\Lambda(t) = \mathcal{G}(t, \Lambda(t)), \tag{12}$$

with the initial condition

$$\Lambda(0) = \Lambda_0, \tag{13}$$

where $\mathcal{G} \in C([0,T] \times \mathbb{R}, \mathbb{R}), 0 < \theta \le 1, \rho > 0$ and ${}^C D_t^{\theta,\rho}$ is the new generalised fractional derivative operator.

Since $\mathcal{G}$ is considered as a continuous mapping so every solution of the IVP (12) is also satisfy the given Volterra integral equation (VIE):

$$\Lambda(t) = \Lambda_0 + \frac{\rho^{1-\theta}}{\Gamma(\theta)} \int_0^t \tau^{\rho-1}(t^\rho - \tau^\rho)^{\theta-1}\mathcal{G}(\tau, \Lambda(\tau))d\tau, \quad t \in [0,T]. \tag{14}$$

Also the IVP(12) is equivalent to the IVP

$$
\begin{aligned}
&{}^{C}D_t^{\theta,\rho}(\Lambda(t)-\Lambda_0)=\mathcal{G}(t,\Lambda(t)),\\
&\Lambda(0)=\Lambda_0.
\end{aligned}
$$

Now first we mention the solution existence of the given IVP by the following theorem.

Theorem 4.1. *[4, 9] (Existence analysis). Let* $0<\theta\leq 1$, $\Lambda_0\in\mathbb{R}$, $\eta>0$ *and* $a>0$. *Let* $R_0:=\{(t,\Lambda):\ t\in[0,a],\ |\Lambda-\Lambda_0|\leq\eta\}$ *and assume that the mapping* $\mathcal{G}:R_0\to\mathbb{R}$ *be continuous. Next, allocate* $S:=\sup_{(t,\Lambda)\in R_0}|\mathcal{G}(t,\Lambda)|$ *and*

$$
T=\begin{cases} a, & if S=0,\\ min\{a,\left(\dfrac{\eta\Gamma(\theta+1)\rho^\theta}{S}\right)^{\dfrac{1}{\theta}}\} & else.\end{cases} \tag{15}
$$

Then, here a function $\Lambda\in\mathcal{C}[0,T]$ *exists that satisfy the IVP* (12) *and* (13).

Theorem 4.2. *Consider the initial value problem proposed by* (12)-(13). *Let*

$$
f(\nu,\Lambda_*(\nu))=\mathcal{G}(t^\rho-(t^\theta-\nu\Gamma(\theta+1)\rho^\theta)^{\frac{1}{\theta}},\Lambda(t^\rho-(t^\theta-\nu\Gamma(\theta+1)\rho^\theta)^{\frac{1}{\theta}})),
$$

with the assumption of Theorem 4.1 *hold. Then, a solution of* (12), $\Lambda(t)$, *is established by*

$$
\Lambda(t)=\Lambda_*(t^\theta\rho^{-\theta}/\Gamma(\theta+1)),
$$

where $\Lambda_*(\nu)$ *is a solution of classical differential equations*

$$
\frac{d\Lambda_*(\nu)}{d\nu}=f(\nu,\Lambda_*(\nu)), \tag{16}
$$

and

$$
\Lambda_*(0)=\Lambda_0. \tag{17}
$$

Proof. The solution of the (12)-(13) is existed from the result of Theorem 4.1. If $\Lambda(t)$ is a solution of (12)-(13) then, it is also satisfy (14). Let $\tau^{\rho} = t^{\rho} - (t^{\theta} - \nu\Gamma(\theta+1)\rho^{\theta})^{1/\theta}$. So, VFIE (14) can be established as

$$\Lambda(t) = \Lambda_0 + \int_0^{t^{\theta}\rho^{-\theta}/\Gamma(\theta+1)} \mathcal{G}(t^{\rho} - (t^{\theta} - \nu\Gamma(\theta+1)\rho^{\theta})^{1/\theta}, \Lambda(t^{\rho} - (t^{\theta} - \nu\Gamma(\theta+1)\rho^{\theta})^{1/\theta}))d\nu$$
$$= \Lambda_0 + \int_0^{t^{\theta}\rho^{-\theta}/\Gamma(\theta+1)} f(\nu, \Lambda_*(\nu))d\nu. \tag{18}$$

Also every solution of (16)-(17) satisfies the VIE given below and vice versa.

$$\Lambda_*(\nu) = \Lambda_0 + \int_0^{\nu} f(s, \Lambda_*(s))ds, \quad 0 \le \nu \le a^{\theta}\rho^{-\theta}/\Gamma(\theta+1). \tag{19}$$

Since $0 \le t^{\theta}\rho^{-\theta}/\Gamma(\theta+1) \le a^{\theta}\rho^{-\theta}/\Gamma(\theta+1)$, the right-hand part of equation (18) is equal to $\Lambda_*(t^{\theta}\rho^{-\theta}/\Gamma(\theta+1))$. $\square$
A simplification of Theorem 4.1 and 4.2 for n-dimensional system is as follows:

Theorem 4.3. *Let $\|.\|$ denotes any convenient norm on R^n. Let $\mathcal{A} \in [R_1, R^n]$, where $R_1 = \{(t,\Lambda) : 0 \le t \le a$ and $|\Lambda - \Lambda_0| \le K\}, \mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n)^T, \Lambda = (\Lambda_1, \Lambda_2, ..., \Lambda_n)^T$, and let $\|\mathcal{A}(t,\Lambda)\| \le M$, on R_1. Then, atleast one solution for the given system of FDEs is exists and defined by*

$$^{C}D^{\theta,\rho}\Lambda(t) = \mathcal{A}(t, \Lambda(t)), \tag{20}$$

with initial conditions

$$\Lambda(0) = \Lambda_0, \tag{21}$$

on $0 \le t \le T^*$, where $T^* = \min\big(a, \big[\frac{K}{M}\Gamma(\theta+1)\rho^{\theta}\big]^{\frac{1}{\theta}}\big),\ 0 < \theta \le 1,\ \rho > 0.$

Theorem 4.4. *Consider the IVP demonstrated by (20)-(21) of order $\theta, 0 < \theta \le 1, \rho > 0$. Assume*

$$f(\nu, \Lambda_*(\nu)) = \mathcal{A}(t^{\rho} - (t^{\theta} - \nu\rho^{\theta}\Gamma(\theta+1))^{1/\theta}, \Lambda(t^{\rho} - (t^{\theta} - \nu\rho^{\theta}\Gamma(\theta+1))^{1/\theta})),$$

then when the Theorem 4.3 hold, a solution of (12)-(13), $\Lambda(t)$, can be expressed by

$$\Lambda(t) = \Lambda_*(t^{\theta}\rho^{-\theta}/\Gamma(\theta+1)),$$

where $\Lambda_*(\nu)$ is a solution of the model of classical differential equations

$$\frac{d\Lambda_*(\nu)}{d\nu} = f(\nu, \Lambda_*(\nu)),$$

alongwith the constraints

$$\Lambda_*(0) = \Lambda_0. \tag{22}$$

5 Important Examples

Here we give some important examples which are interpreted by utilizing Mathematica 10 software.

Example 4.1 Consider the following linear non-homogeneous non-integer order equation

$$\begin{aligned} &\mathcal{D}^{\theta,\rho}_{0+} y(t) = t, \;\; \rho > 0, \;\; 0 < \theta \leq 1, \\ &y(0) = y_0. \end{aligned} \tag{23}$$

For this example,

$$g(\nu) = \frac{\rho^{-\theta}\left(\theta\big((\theta+1)t^{\theta} - \nu\rho^{\theta}\Gamma(\theta+2)\big)\big(t^{\theta} - \nu\rho^{\theta}\Gamma(\theta+1)\big)^{\frac{1}{\theta}} + (\theta+1)\left(\rho^{\theta}\Gamma(\theta+2)(\nu t^{\rho} + y_0) - \theta\big(t^{\theta}\big)^{\frac{1}{\theta}+1}\right)\right)}{(\theta+1)\Gamma(\theta+2)}.$$

The solution of the related integer order problem as mentioned in Theorem 4 is

$$y_1(\nu) = \frac{\rho^{-\theta}t^{\theta}\left((\theta+1)t^{\rho} - \theta\big(t^{\theta}\big)^{\frac{1}{\theta}}\right)}{\Gamma(\theta+2)} + y_0.$$

So, the solution of the given non-integer order problem is

$$y(t) = y_1\left(\frac{\rho^{-\theta}t^{\theta}}{\Gamma(\theta+1)}\right) = \frac{\rho^{-\theta}t^{\theta}\left((\theta+1)t^{\rho} - \theta\left(t^{\theta}\right)^{\frac{1}{\theta}}\right)}{\Gamma(\theta+2)} + y_0. \tag{24}$$

Indeed, it can be illustrated that (24) is a solution of (23) by using the generalized non-integer order derivative. it should be note that solution

(24) is the same as solution [p.2757, Eq.(11)] obtained in [3] for $\theta = 1/2$ and $\rho = 1$.

Example 4.2 Let us remind the fractional order Riccati differential equation given in [16]:

$$\begin{aligned} &\mathcal{D}_{0+}^{\theta,\rho} y(t) = 2y(t) - y^2(t) + 1, \quad t, \ \rho > 0, \quad 0 < \theta \leq 1, \\ &y(0) = 0. \end{aligned} \tag{25}$$

The exact solution of (25), at $\rho = 1$ and $\theta = 1$ is

$$y(t) = 1 + \sqrt{2} \tanh \left[\sqrt{2} t + \frac{1}{2} \log \left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1} \right) \right].$$

In Fig. 1, we give the comparison of the approximate solution plots of Eq.(25) by the new method and the adaptive predictor-corrector algorithm [16] with the exact solution of Eq.(25). Here we checked that the approximate solutions are in contract with exact solution, graphically. The absolute errors in the proposed techniques are given in Figure 2 where solid-type and dashed-type lines demonstrate errors in the new method and the adaptive predictor-corrector algorithm, respectively.

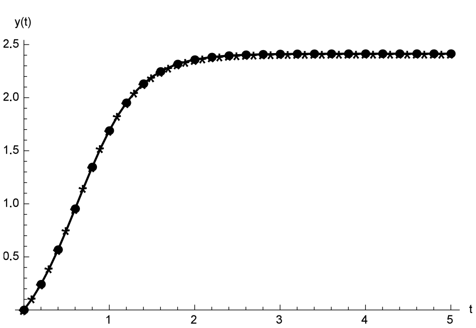

Figure 1: Solutions of equation (25)

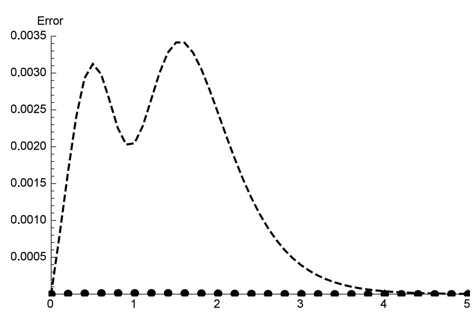

Figure 2: Error in approximate solutions of (25)

Figure 3

Example 4.3 Now we choose the following non-classical order differential equation system given in [2]:

$$\begin{aligned}
&\mathcal{D}_{0+}^{\theta,\rho}x(t) = wx(t) - y^2(t),\\
&\mathcal{D}_{0+}^{\theta,\rho}y(t) = \mu(z(t) - y(t)),\\
&\mathcal{D}_{0+}^{\theta,\rho}z(t) = ay(t) - bz(t) + x(t)y(t),
\end{aligned} \tag{26}$$

where $t,\ \rho > 0,\ \ 0 < \theta \leq 1$, and $a,\ b,\ w,\ \mu$ are constant quantities. Moreover, $w = -2.667,\ a = 27.3,\ b = 1,\ \mu = 10$ (time step $h = 0.02$, initial conditions are $(0, 10, 10)$).

Figures 4, 5 and 7 show the solutions $x(t),\ y(t)$ and $z(t)$ of the system (26) for $(\theta = 0.89,\ \rho = 1.2)$ whereas 8, 10 and 11 show phase portrait of the system (26) for the same values of θ and ρ. The CPU time, needed to get the solution for the system (26) by the new method is just 0.265625 in seconds. It may be observed from Figures 8, 10 and 11 that the system show chaotic behaviour for $(\theta = 0.89,\ \rho = 1.2)$.

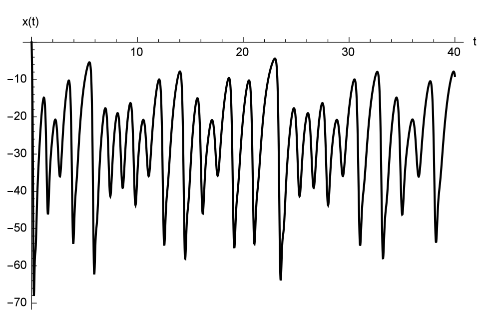

Figure 4: Outputs of Eqn. (26) for in the (t, x)-plane.

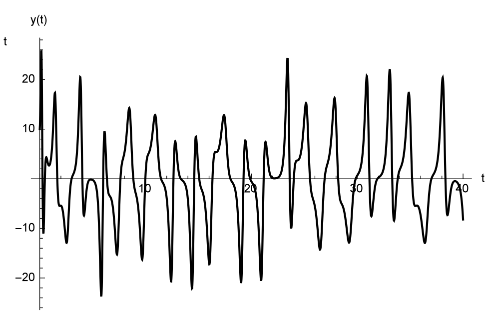

Figure 5: Outputs of Eqn. (26) for in the (t, y)-plane.

Figure 6

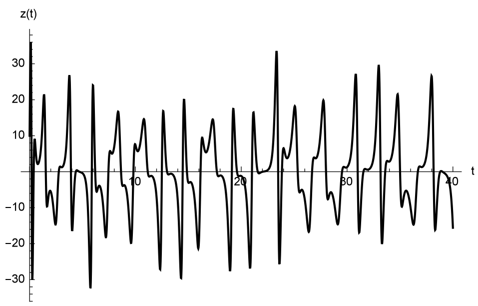

Figure 7: Outputs of Eqn. (26) for in the (t, z)-plane.

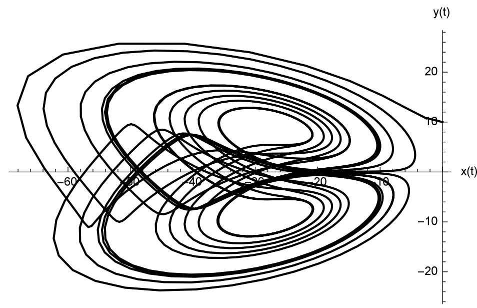

Figure 8: Outputs of Eqn. (26) in the xy-phase plane.

Figure 9

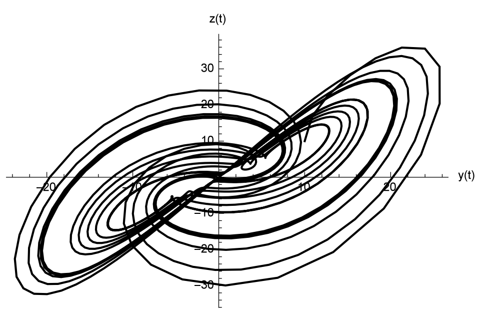

Figure 10: Solutions of equation (26) in the yz-phase plane.

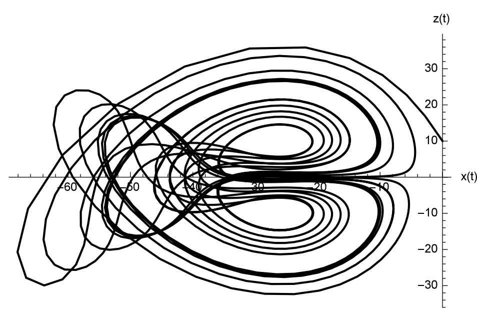

Figure 11: Solutions of equation (26) in the xz-phase plane.

Figure 12

Example 4.4 As a last example, we consider a computer virus model proposed by *Piqueira et al.* [18] in integer order sense, which is as follows:

$$
\left\{
\begin{aligned}
\frac{dS}{dt} &= -\beta_{SA} SA - \alpha SI + \eta R, \\
\frac{dI}{dt} &= -\beta_{IA} IA + \alpha SI - \zeta I, \\
\frac{dR}{dt} &= -\eta R + \zeta I, \\
\frac{dA}{dt} &= \beta_{SA} SA + \beta_{IA} IA,
\end{aligned}
\right. \tag{27}
$$

where $S(t)$ is for susceptible computers subjected to possible infection, $A(t)$ for non-infected computers furnished with anti-virus, $I(t)$ denotes virus-infected computer systems and $R(t)$ define removed computers due to infection or not. Parameter β_{SA} denotes the conversion rate of susceptible into antidotal, α denotes the transmission rate of susceptible into infection, β_{IA} is the transmission rate of infected computers into antidotal, ζ is the removed rate and η is the rate of removed computers which can be restored and varied into susceptible.
Now for numerical simulations we use two different values of given parameters with initial conditions for endemic equilibrium (EE) and disease-free equilibrium (DFE) conditions, respectively as given in [18] .
Numerical values for DFE case;
$\alpha = 0.1,\ \zeta = 20,\ \beta_{SA} = 0.025,\ \beta_{IA} = 0.25,\ \eta = 0.8$ with initial constraints $S(0) = 74,\ I(0) = 25,\ R(0) = 0,\ A(0) = 1.$
Numerical values for EE case;
$\alpha = 0.1,\ \zeta = 9,\ \beta_{SA} = 0.025,\ \beta_{IA} = 0.25,\ \eta = 0.8$ with initial restrictions $S(0) = 3,\ I(0) = 95,\ R(0) = 1,\ A(0) = 1.$
Now the generalization of the above system in the new modified Caputo-type non-classical derivative sense is as follows:

$$\begin{cases} {}^{C}D_t^{\theta,\rho} S = -\beta_{SA} SA - \alpha SI + \eta R, \\ {}^{C}D_t^{\theta,\rho} I = -\beta_{IA} IA + \alpha SI - \zeta I, \\ {}^{C}D_t^{\theta,\rho} R = -\eta R + \zeta I, \\ {}^{C}D_t^{\theta,\rho} A = \beta_{SA} SA + \beta_{IA} IA, \end{cases} \tag{28}$$

The related integer order system given in Theorem 4.2 is

$$\begin{cases} \dfrac{dS^*}{dt} = -\beta_{SA} S^* A^* - \alpha S^* I^* + \eta R^*, \\ \dfrac{dI^*}{dt} = -\beta_{IA} I^* A^* + \alpha S^* I^* - \zeta I^*, \\ \dfrac{dR^*}{dt} = -\eta R^* + \zeta I^*, \\ \dfrac{dA^*}{dt} = \beta_{SA} S^* A^* + \beta_{IA} I^* A^*, \end{cases} \tag{29}$$

If $(S_*(\nu),\ I_*(\nu),\ R_*(\nu),\ A_*(\nu)$, is the solution of this integer-order model then the solution of the system (28) is $(S_*(t^\theta \rho^{-\theta}/\Gamma(\theta+1)),\ I_*(t^\theta \rho^{-\theta}/\Gamma(\theta+$

1)), $R_*(t^\theta \rho^{-\theta}/\Gamma(\theta+1))$, $A_*(t^\theta \rho^{-\theta}/\Gamma(\theta+1)))$. So the numerical solution of the system (28) for the given numerical values is calculated using the method of Theorem 4.2.

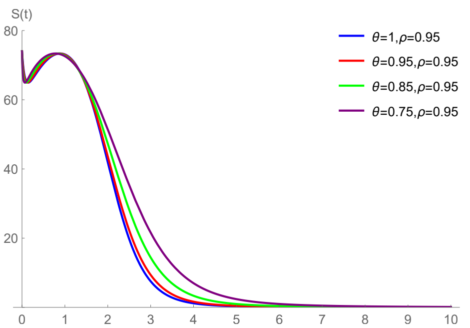

Figure 13: Nature of $S(t)$ for DFE case

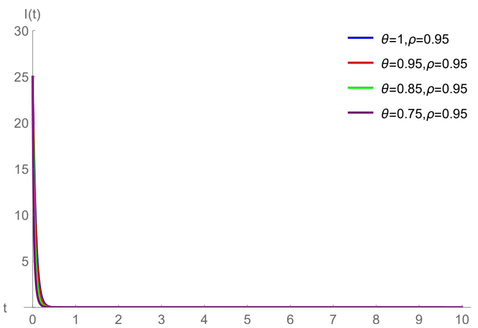

Figure 14: Nature of $I(t)$ for DFE case

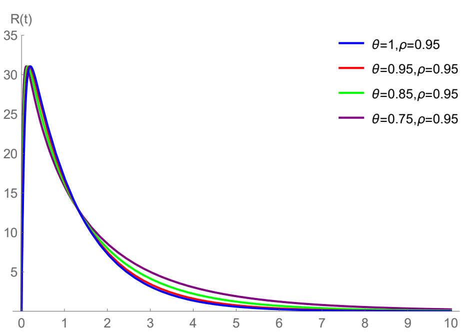

Figure 15: Nature of $R(t)$ for DFE case

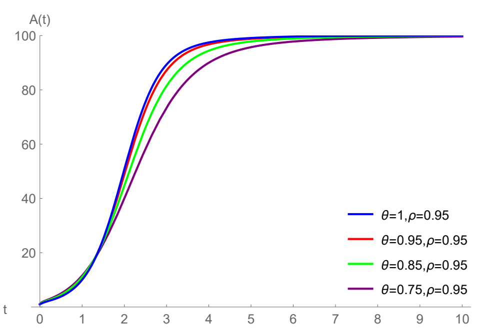

Figure 16: Nature of $A(t)$ for DFE case

Figure 17: Nature of all given classes for DFE case

From the above graphical simulations, we studied the nature of $S(t)$ susceptible, non-infected $A(t)$, infected $I(t)$ and removed $R(t)$ computers in Figures 13, 14, 15 and 16 respectively. In the group of Figure 17, we settled the parameter values for DFE case.

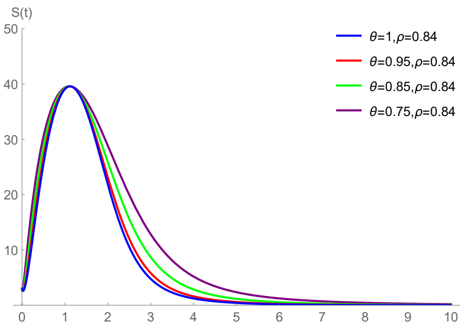

Figure 18: Nature of $S(t)$ for EE case

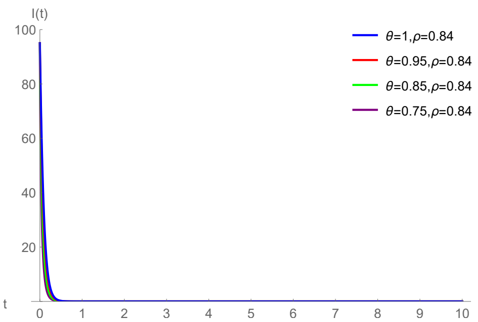

Figure 19: Nature of $I(t)$ for EE case

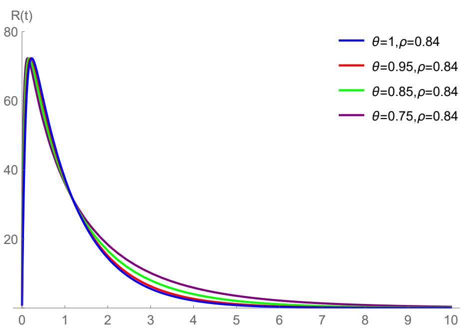

Figure 20: Nature of $R(t)$ for EE case

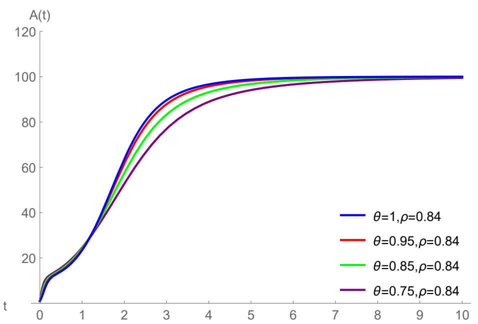

Figure 21: Nature of $A(t)$ for EE case

Figure 22: Nature of all given classes for EE case

To perform the simulations for EE case, we studied the nature of $S(t)$ susceptible, non-infected $A(t)$, infected $I(t)$ and removed $R(t)$ computers in Figures 18, 19, 20 and 21 respectively. In the group of Figure 22, we settled the parameter values for EE case. From the all above graphical calculations, we concluded that the given numerical technique works well to frame the structures of non-linear epidemic models.

6 Conclusions

In this article, we have proposed an algorithm for finding the solution of non-linear new generalized Caputo type non-classical differential equations of order $0 < \theta \leq 1$. We have proposed a classical differential equation with the help of the given FDEs and given the propinquity between their solutions. A generalized form of the given technique to finite systems is also presented. By this given scheme, we can find the exact solutions of the important FDEs in the terms of classical differential equations solution, which is the main benefit of the proposed scheme. This technique is precious in the applications of FDEs in various fields. There are so many various types of numerical techniques are available to find the solutions of applied fractional order problems but the techniques for integer-order equations are much stronger in the view of chastity and convergence rate. By this scheme, we can use the numerical techniques for classical differential equations for the numerical solutions of non-integer order differential equations. Some important examples are explained with the comparison of their exact solutions to the numerical solutions founded by some other techniques. Solution of a computer virus epidemic model is also given to prove the availability of the proposed method in mathematical epidemiology. We hope that the given technique will become useful to solve some important FDEs and easy in the implementations.

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Conflict of interest

There is no conflicts of interest exist in this paper. No Funding is Received for this study.

Author's Contribution

Pushpendra Kumar: Formal analysis, Conceptualization, Methodology, Visualization, Investigation, Resources, Writing - original draft.
Vedat Suat Erturk: Software, Investigation, Conceptualization, Visualization, Writing- review & editing.
Anoop Kumar: Writing-review & editing.
Mustafa Inc: Writing-review & editing.

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